LEARNING SEMINAR ON CHROMATIC HOMOTOPY THEORY

ABSTRACT. The goal of (most) homotopy theorists is to compute the stable homotopy groups of spheres. However, this is (very) hard. Recent developments in chromatic homotopy theory and higher algebra gave new insights about the structure of homotopy groups of spheres. It originates from Quillen's work on the relation between complex cobordism groups and homotopy groups of spheres.

FIGURE 1. Sonia DELAUNAY, Automne, 1965, lithographie en couleurs, collection musée des Beaux-Arts de Brest.

Talks 10 to 13 will be decided later, based on participants' preferences. Feel free to suggest any topic you would like to cover, either in-person or via email. Please find below a list of potential topics we could cover:

- Chromatic Nullstellensatz
- Galois Descent for $K(1)$-localized K-theory (Thomason's paper)
- Telescope conjecture
- Power Operations
- Multiplicative structures on Moore spectra
- A chromatic approach to homological stability (a bit of a stretch)
- McClure's Theorem
- Brown-Peterson spectra
- Redshift/Blueshift phenomena

1. Overview and Background

Speaker: Carlos Andrés Alvarado Álvarez

Abstract: Short introduction to chromatic homotopy theory by saying goal of life is to compute homotopy groups of spheres + do a tower approximating $\mathbb{S}$. Spectra as Ω-spectra, Brown representability and cohomology theories, basic features of Sp such as symmetric monoidality, stable. Stable homotopy groups. Loop-suspension adjunction. Examples: suspension spectra, sphere spectrum, $\mathrm{H}\mathbb{F}_p$. Category of E-modules. Localization at a spectrum E. If time permits, define ring spectra.

References:

- Higher Algebra, Lurie
- Stable Homotopy Theory, Adams

2. The complex cobordism ring

Speaker: TBD

Abstract: Define complex K-theory spectrum ku, define MU and relation to complex cobordisms groups, map $\mathbb{S} \to \mathrm{MU}$, complex oriented cohomology theories, Lazard Ring and Quillen's Theorem for $\pi_*(MU)$, explain how the first Chern/Euler class for a complex-oriented cohomology theory gives a formal group law.

References:

- Lurie, Lectures 4 to 7 of https://www.math.ias.edu/~lurie/252x.html
- For complex cobordism, one may look at Milnor-Stasheff book on characteristic classes

3. Formal Group Laws

Speaker: TBD

Abstract: A formal group law can be associated to any complex-oriented cohomology theory, determined by the Euler class. Define Lazard ring L as a universal formal group law and then state Lazard's and Quillen's theorems. Discuss classification of FGL in char 0 (resp. char p) and introduce the height of FGL.

References:

- Lurie, https://www.math.ias.edu/~lurie/252x.html, lectures $11-12$

4. Moduli stack of Formal Group Laws

Speaker: TBD

Abstract: Define the moduli stack of Formal Group Laws and relevant concepts such as the stratification induced from height, and flat modules

References:

- Lurie, https://www.math.ias.edu/~lurie/252x.html, lectures $13-15$

5. Landweber Exactness and Lubin-Tate Theories

Speaker: Rafah Hajjar Muñoz

Abstract: A flat map Spec(R) $\to \mathcal{M}_{FGL}$ gives a homology theory and Landweber exactness gives conditions for a module M over L to be flat over the stack. Explain what goes into this and then explain Lubin-Tate theories as deformations in $\mathcal{M}_{FGL}$ and build Morava E-theories.

References:

- Lurie, https://www.math.ias.edu/~lurie/252x.html, lecture 16

6. Morava K-theory spectra and nilpotence

Speaker: Azélie Picot

Abstract (follows the Euro Talbot 2025): State the Devinatz–Hopkins–Smith nilpotence theorem. Define Morava K-theory spectra $K(n)$ as in Lecture 22 of Lurie's notes. Discuss the elements v_n. Explain how $K(n)$ detects nilpotence, comment on $K(n)$ being the *only fields in higher algebra*. State the Hopkins-Smith thick subcategory theorem. If time permits, define telescopes $T(n)$ and state the Telescope conjecture, which was recently disproved.

References:

- Lurie, https://www.math.ias.edu/~lurie/252x.html, Lecture 22-27
- Talbot notes by Qi Zhu, https://1429cecd-24a0-4223-8b7c-1ebf47aa12e2.filesusr.com/ugd/8e912a_214e47a1dc284d6991288a90b4047564.pdf
- The papers by Devinatz–Hopkins–Smith [2] and Hopkins–Smith [8]
- Hopkins' survey [6]

7. Descendability, the smash product theorem, and chromatic convergence

Speaker: Sangmin Ko

Abstract (follows the Euro Talbot 2025): Discuss the smash product theorem. Introduce descendable objects, nilpotent and quickly converging towers and apply these notions to sketch the proof of the smash product theorem. Discuss the chromatic convergence theorem.

References:

- Mathew's paper [10] for the general notions and a brief discussion of the smash product theorem (Section 3.5. of loc. cit.)
- Ravenel's orange book [11] for the smash product theorem
- Lurie's lecture notes: https://www.math.ias.edu/~lurie/252x.html
- Talbot notes by Qi Zhu, https://1429cecd-24a0-4223-8b7c-1ebf47aa12e2.filesusr.com/ugd/8e912a_214e47a1dc284d6991288a90b4047564.pdf

8. Image of J and the $K(1)$-local sphere

Speaker: Azélie Picot

Abstract: Chromatic convergence theorem gives a tower which converges to $\mathbb{S}$, goal of this talk is to describe the first stage of the tower: $\mathrm{L}_{\mathrm{E}(1)}\mathbb{S}$. Define J-homomorphism, state how $\mathrm{L}_{\mathrm{E}(1)}\mathbb{S}$ determined by image of J-homomorphism, describe $\mathrm{L}_{\mathrm{K}(1)}\mathbb{S}$, state chromatic fracture square, use it to deduce $\mathrm{L}_{\mathrm{E}(1)}\mathbb{S}$ from $L_{K(1)}\mathbb{S}$.

References:

- Lurie, https://www.math.ias.edu/~lurie/252xnotes/Lecture35.pdf, Lecture 35

9. Special features of monochromatic homotopy theory

Speaker: Carlos Andrés Alvarado Álvarez

Abstract(follows the Euro Talbot 2025): Monochromatic categories (i.e. $K(n)$-local and $T(n)$-local spectra) are strange. Firstly, Bousfield-Kuhn (BK) functors Φ_n which says $K(n)$-localization factors through spaces. Construction. If time permits, sketch of proof. Secondly, consequence of BK functor is that the Tate construction vanishes in $\mathrm{Sp}_{T(n)}$. Define the notion of higher semi-additivity and give examples. The $K(n)$-local and $T(n)$-local categories are ∞-semi-additive.

References:

- Talbot Notes by Qi Zhu https://1429cecd-24a0-4223-8b7c-1ebf47aa12e2.filesusr.com/ugd/8e912a_214e47a1dc284d6991288a90b4047564.pdf
- Generalities on the Bousfield–Kuhn functors: Kuhn's survey [9] or the exposition in [4].
- Ambidexterity: the K(n)-local case is due to Hopkins–Lurie [7], the T(n)-local case to Carmeli–Schlank–Yanovski [1].
- Clausen–Mathew give a very short proof of Tate vanishing directly from the existence of the Bousfield–Kuhn functors.
- Unstable v_n-periodic homotopy theory, see [3], [5], and [4].

10. TBD

11. TBD

12. TBD

13. TBD

References

1. Shachar Carmeli, Tomer M. Schlank, and Lior Yanovski, *Ambidexterity in chromatic homotopy theory*, 2020.
2. Ethan S. Devinatz, Michael J. Hopkins, and Jeffrey H. Smith, *Nilpotence and stable homotopy theory i*, Annals of Mathematics **128** (1988), no. 2, 207–241.
3. Rosona Eldred, Gijs Heuts, Akhil Mathew, and Lennart Meier, *Monadicity of the bousfield-kuhn functor*, 2017.
4. Gijs Heuts, *Lie algebra models for unstable homotopy theory*, 2019.
5. ———, *Lie algebras and v_n-periodic spaces*, 2020.
6. Michael J. Hopkins, *Global methods in homotopy theory*, London Mathematical Society Lecture Note Series, p. 73–96, Cambridge University Press, 1987.
7. Michael J. Hopkins and Jacob Lurie, *Ambidexterity in $k(n)$-local stable homotopy theory*, 2013.
8. Michael J. Hopkins and Jeffrey H. Smith, *Nilpotence and stable homotopy theory ii*, Annals of Mathematics **148** (1998), no. 1, 1–49.
9. Nicholas J. Kuhn, *A guide to telescopic functors*, 2008.
10. Akhil Mathew, *Examples of descent up to nilpotence*, 2017.

11. Douglas C. Ravenel, *Nilpotence and periodicity in stable homotopy theory*, 128, Princeton University Press, 1992.